BiiPS

a software for **B**ayesian **i**nference with **i**nteracting **P**article **S**ystems

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Summary

Summary

Bayesian inference

Bayesian inference involves the approximation of a probability distribution of an unknown parameter $x \in E$ given the observations y:

$$\pi = p(x|y)$$

Bayes rule:

$$\begin{aligned} p(x|y) &= \frac{p(x,y)}{p(y)} \\ &= \frac{p(x)p(y|x)}{p(y)} \\ &= \frac{\gamma(x)}{Z} \end{aligned} \tag{1}$$

Marginal likelihood:

$$Z = p(y) = \int_E p(y|x)p(x)dx$$

Bayesian inference

Delivers both an estimate and the associated uncertainty

$$\widehat{x}^{\mathrm{MMSE}} = \mathbb{E}[x|y] = \int_E x \; p(x|y) dx$$

$$Var(\widehat{x}^{\mathrm{MMSE}}) = \mathbb{E}[(x-\widehat{x})^2|Y] = \int_E (x-\widehat{x})^2 \; p(x|y) dx$$

More generally, we can integrate any test function $\varphi(x)$

$$I = \mathbb{E}[\varphi(x)|y] = \int_E \varphi(x) \; p(x|y) dx$$

Bayesian inference

Delivers both an estimate and the associated uncertainty

$$\begin{aligned}\widehat{x}^{\mathrm{MMSE}} &= \mathbb{E}[x|y] \\ &= \int_E x\ p(x|y)dx\end{aligned}$$

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$$I = \mathbb{E}[\varphi(x)|y] = \int_E \varphi(x)\ p(x|y)dx$$

Graphical Models / Bayesian networks

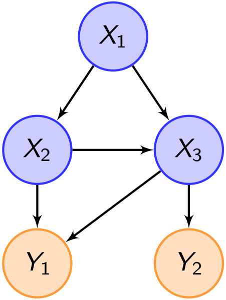


Figure: Directed acyclic graph

The graph displays a factorization of the joint distribution:

$p(x_{1:3}, y_{1:2})$

Graphical Models / Bayesian networks

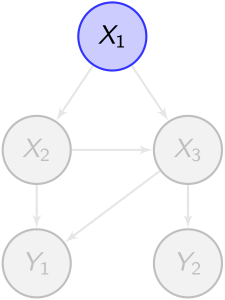
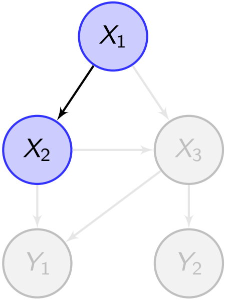


Figure: Directed acyclic graph

The graph displays a factorization of the joint distribution:

$$p(x_{1:3}, y_{1:2}) = p(x_1)\ p(x_2|x_1)\ p(x_3|x_2, x_1)\ p(y_1|x_2, x_3)\ p(y_2|x_3)$$

Graphical Models / Bayesian networks



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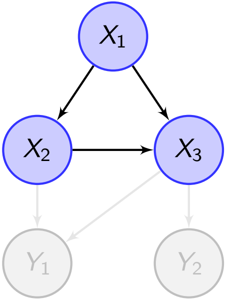

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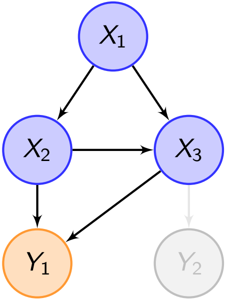

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Graphical Models / Bayesian networks

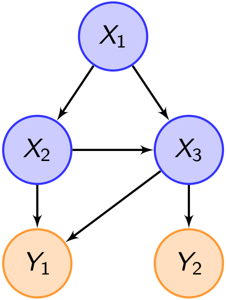

Figure: Directed acyclic graph

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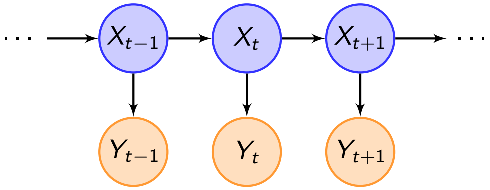
Hidden Markov Models / state-space models



$p(x_0)$ Initial distribution

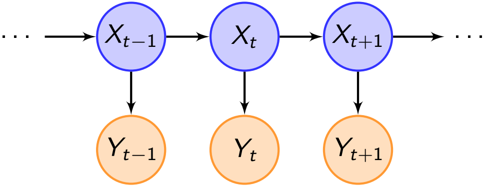
$\forall t = 1, \dots, T$

$$\begin{cases} p(x_{t+1}|x_t) & \text{Evolution model} \\ p(y_t|x_t) & \text{Measurement model} \end{cases}$$

where $x_t \in \mathcal{X}$, e.g. $\mathbb{R}^n$

Bayesian inference

on HMM



$$
\begin{aligned}
X &= X_{0:t} \in E = \mathcal{X}^{t+1} \\
Y &= Y_{1:t}
\end{aligned}
$$

(1) becomes

$$
\begin{aligned}
p(x_{0:t}|y_{1:t}) &= \frac{p(x_{0:t})\ p(y_{1:t}|x_{0:t})}{p(y_{1:t})} \\
&= p(x_{0:t-1}|y_{1:t-1})\ \frac{p(x_t|x_{t-1})\ p(y_t|x_t)}{p(y_t|y_{1:t-1})}
\end{aligned}
$$

Bayesian inference

- Many problems (inverse problems, filtering, tracking, deconvolution, etc..) can be formulated in this context
- The posterior distribution is usually not calculable analytically
  - complex non linear models
  - high dimension
- ... hence requiring the use of stochastic simulation techniques

Bayesian inference

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  - high dimension
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BUGS

Bayesian inference Using Gibbs Sampling

- More than 20 years history
  - 'Classic' BUGS: started in 1989 at MRC Biostatistics Unit
  - WinBUGS: co-developped with Imperial College School of Medicine
  - OpenBUGS: open source release and experimental development
  - JAGS © Martyn Plummer: open source clone in C++
- Expert system runs MCMC methods (Gibbs, Metropolis, ...) in a 'black-box' fashion
  - Iterative algorithms: approximately sampling according to target *posterior* distribution
- User-friendly
- Very popular among practitioners, applying MCMC methods to a wide range of applications

Summary

Particle methods

- A new generation of stochastic algorithms has emerged in recent years (particle filtering, sequential Monte Carlo methods, etc.).
- Based on interacting particle systems
- Two stochastic mechanisms
  1. Mutation: the particles explore their environment randomly and independently
  2. Selection: the best suited particles are duplicated, others removed

- Designed to sample from a sequence of distributions $\pi_k(x_{1:k})$ *e.g.* $\pi_k(x_{1:k}) = p(x_{1:k}|y_{1:k})$
- when we can only compute the unnormalized version $\gamma_k(x_{1:k})$

$$\begin{aligned}\pi_k(x_{1:k}) &= \frac{\gamma_k(x_{1:k})}{Z_k} \\ &= \frac{p(x_{1:k}, y_{1:k})}{p(y_{1:k})}\end{aligned}$$

Particle methods

- A new generation of stochastic algorithms has emerged in recent years (particle filtering, sequential Monte Carlo methods, etc.).
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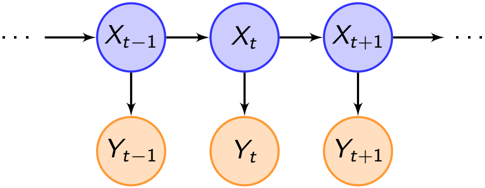
Example



1D linear gaussian state-space model

$$X_0 \sim \mathcal{N}(0, 1)$$

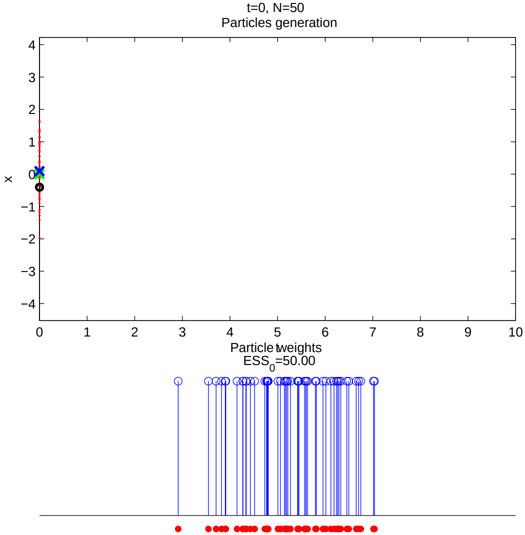
Pour $t = 1, \dots, 20$

$$X_t | X_{t-1} \sim \mathcal{N}(X_{t-1}, 1)$$

$$Y_t | X_t \sim \mathcal{N}(X_t, 2)$$

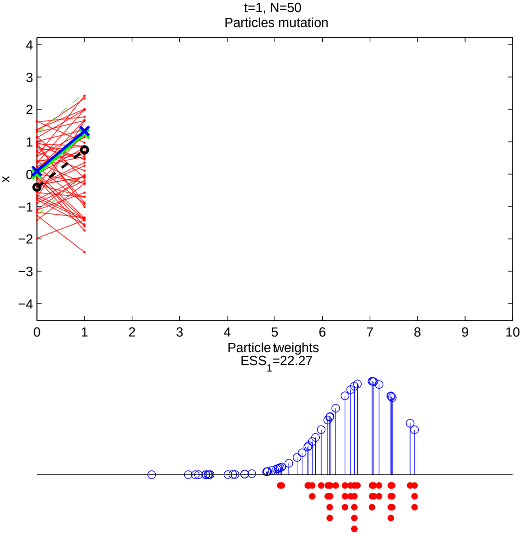
- Filtering problem: estimate $p(x_t | y_{1:t})$
- Optimal solution tractable by Kalman filter

Example



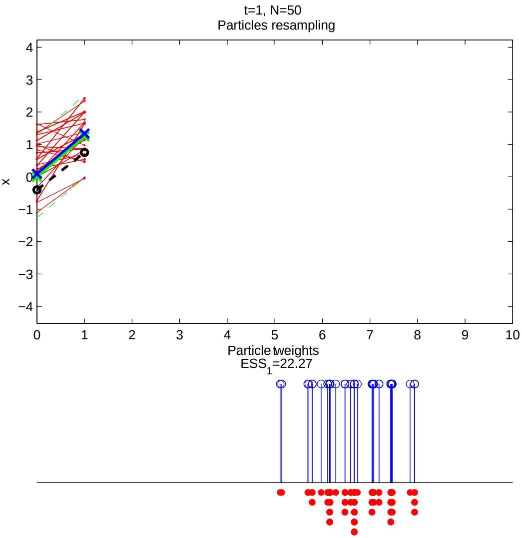
t=0, N=50
Particles generation
x
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
9
10
Particle weights
ESS$_0$=50.00

Example

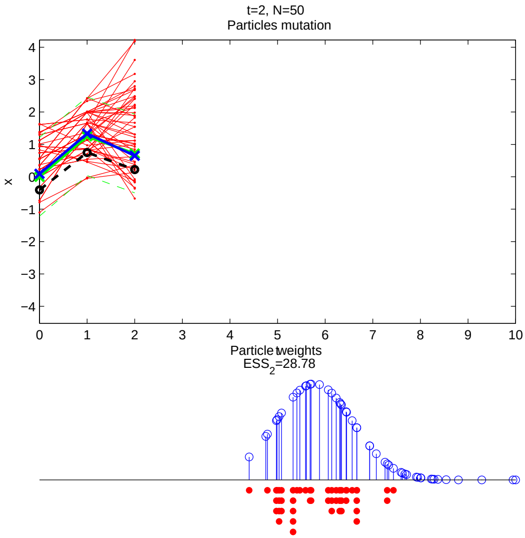


t=1, N=50
Particles mutation
x
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
9
10
Particle weights
ESS_1=22.27

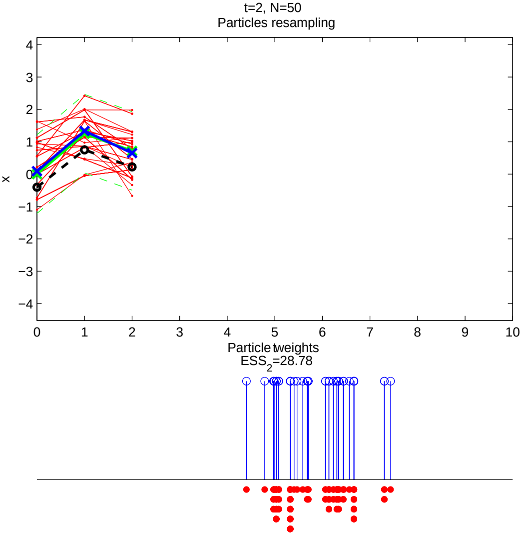
Example



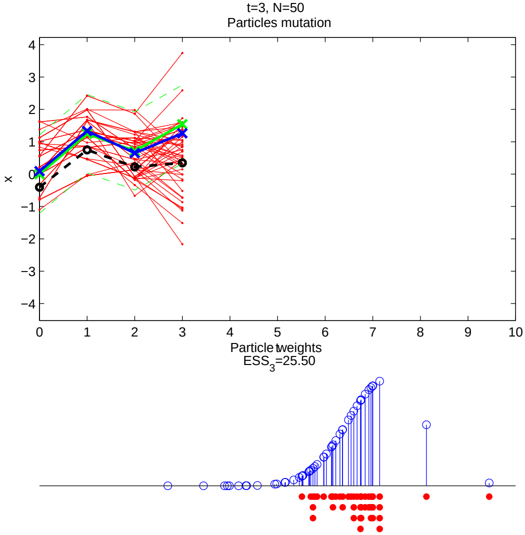
Example



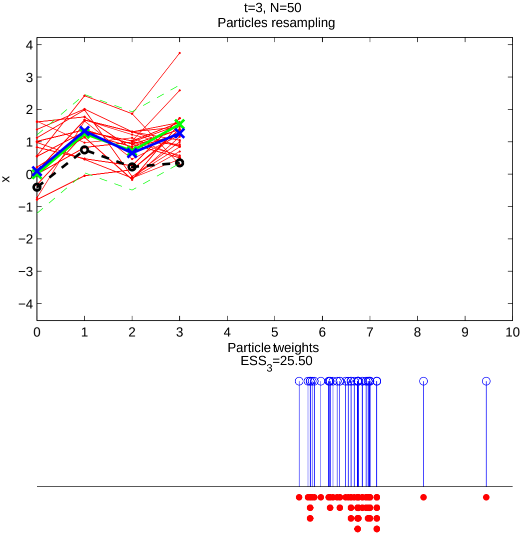
Example


t=2, N=50
Particles resampling
x
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
9
10
Particle weights
ESS$_2$=28.78

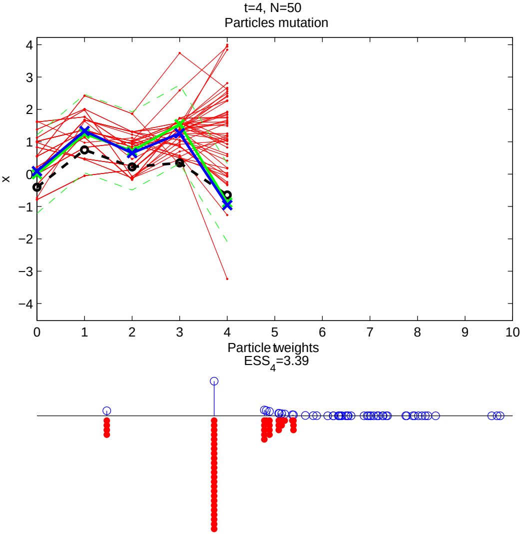
Example



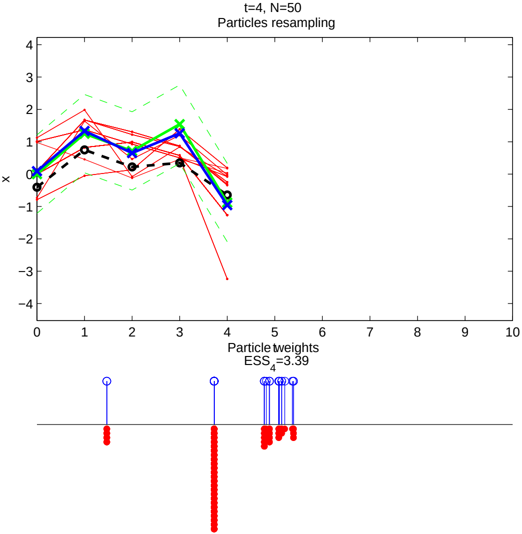
Example


t=3, N=50
Particles resampling
4
3
2
1
0
−1
−2
−3
−4
x
0
1
2
3
4
5
6
7
8
9
10
Particle weights
ESS_3=25.50

Example

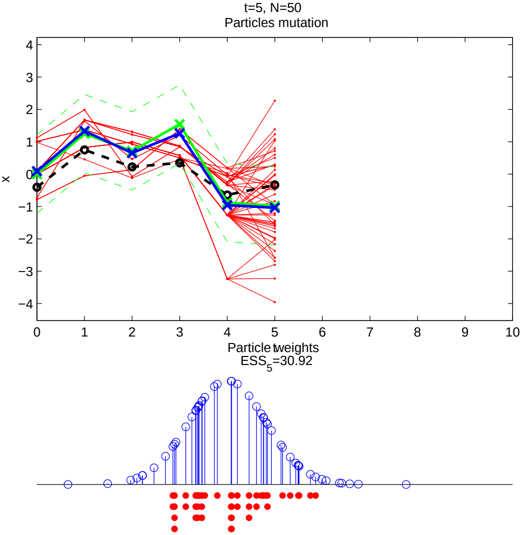


Example

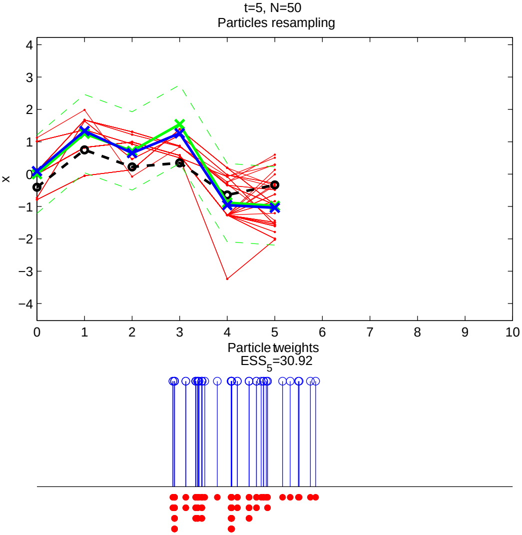


t=4, N=50
Particles resampling
x
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
9
10
Particle weights
ESS$_4$=3.39

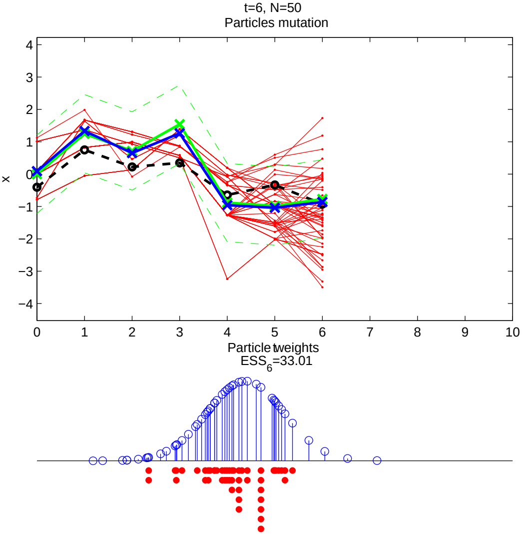
Example


t=5, N=50
Particles mutation
x
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
9
10
t
Particle weights
ESS$_5$=30.92

Example

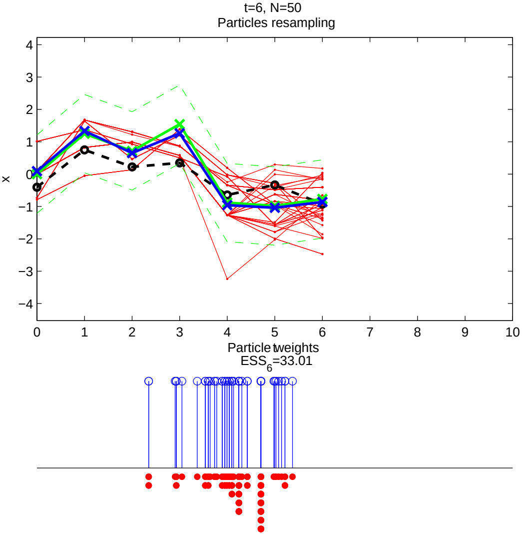


Example



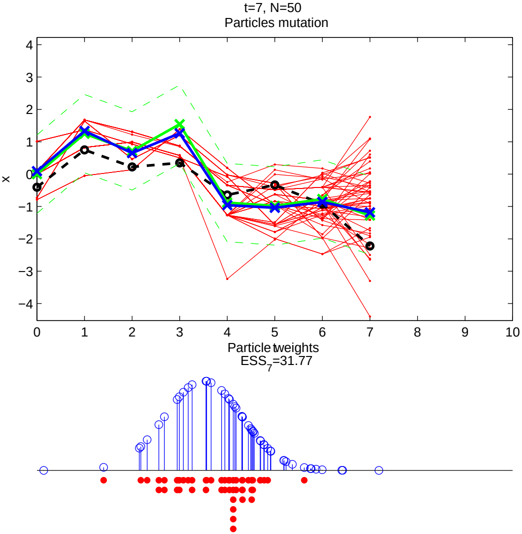
t=6, N=50
Particles mutation
x
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
9
10
Particle weights
ESS_6=33.01

Example



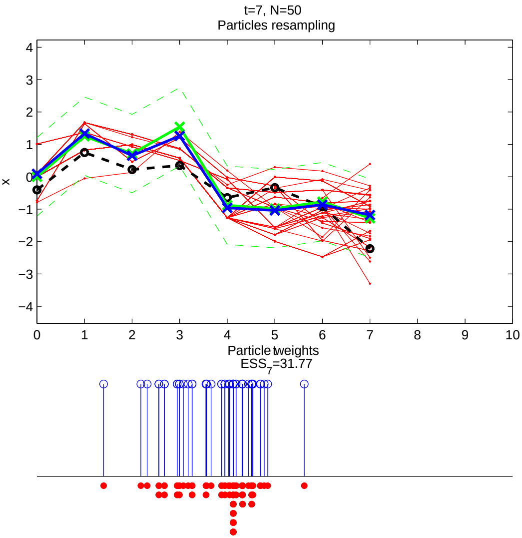
t=6, N=50
Particles resampling
x
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
9
10
t
Particle weights
ESS_6=33.01

Example



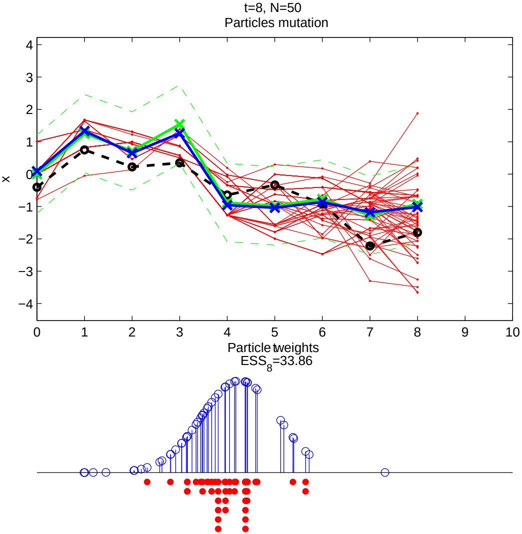
t=7, N=50
Particles mutation
x
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
9
10
Particle weights
ESS$_7$=31.77

Example



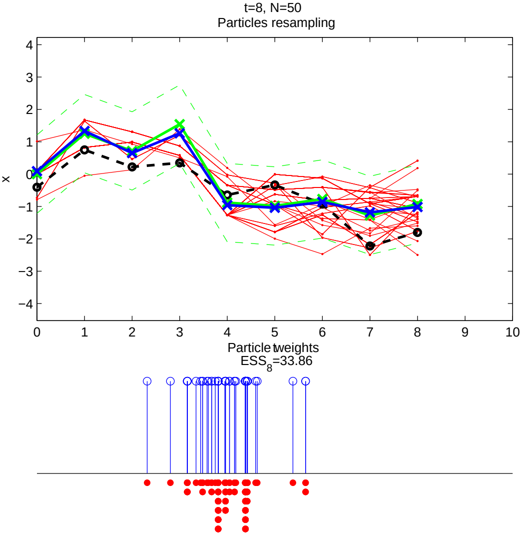
t=7, N=50
Particles resampling
x
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
9
10
Particle weights
ESS7=31.77

Example



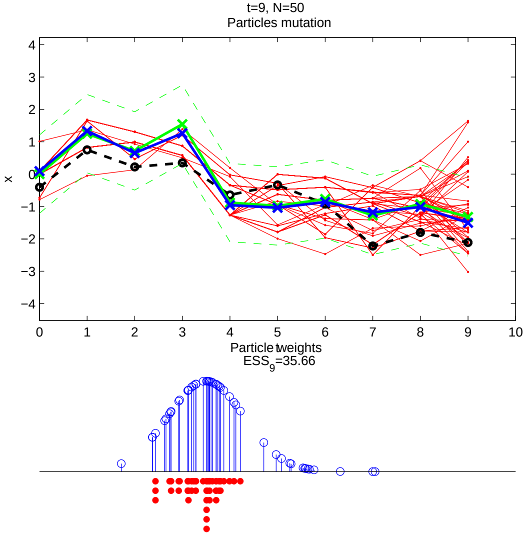
t=8, N=50
Particles mutation
x
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
9
10
Particle weights
ESS_8=33.86

Example

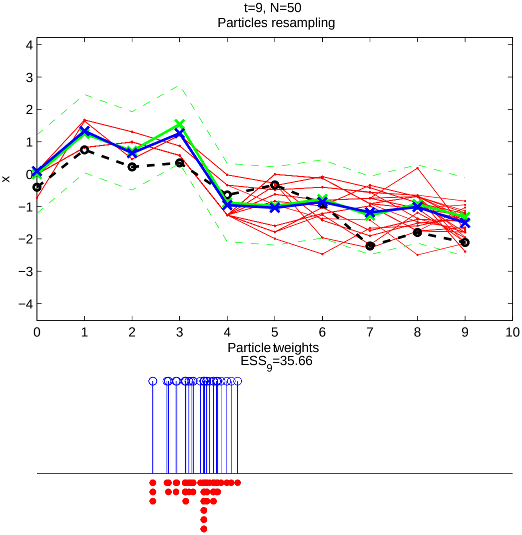


t=8, N=50
Particles resampling
x
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
9
10
Particle weights
ESS$_8$=33.86

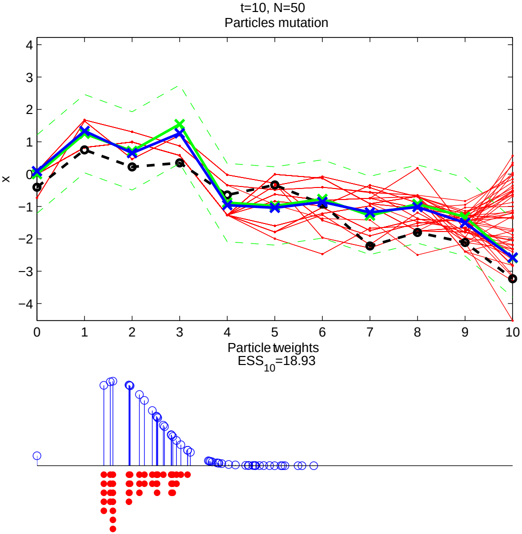
Example



Example


t=9, N=50
Particles resampling
x
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
9
10
Particle weights
ESS$_9$=35.66

Example


t=10, N=50
Particles mutation
x
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
9
10
Particle weights
ESS10=18.93

Particle methods

Generic SMC algorithm

SMC algorithm with N particles

- At time 1: for $i = 1, \dots, N$
  - Sample $x_1^{(i)} \sim q_1(x_1)$
  - Compute unnormalized weights $w_1^{(i)} = \frac{\gamma(x_1^{(i)})}{q_1(x_1^{(i)})}$
- At time $k = 2, \dots, n$: for $i = 1, \dots, N$
  - Resample $\{x_{k-1}^{(i)}, W_{k-1}^{(i)}\}$ and set $W_{k-1}^{(i)} = 1/N$
  - Sample $x_k^{(i)} \sim q_k(x_k | x_{1:k-1})$
  - Compute unnormalized weights $w_k^{(i)} = W_{k-1}^{(i)} \underbrace{\frac{\gamma_k(x_{1:k}^{(i)})}{\gamma_{k-1}(x_{1:k-1}^{(i)}) q_k(x_k^{(i)})}}_{\text{Incremental weight}}$

Particle methods

Estimates

- At time k we can approximate integrals $I_k = \mathbb{E}_{\pi_k}[\varphi(x_k)]$ by

$$\widehat{I_k} = \sum_{i=1}^{N} W_k^{(i)} \varphi(x_n^{(i)})$$

- We also obtain sequential approximations of the normalizing constant

$$Z_k = Z_{k-1} \frac{1}{N} \sum_{i=1}^{N} \alpha_k(x_{1:k}^{(i)})$$

- Effective Sample Size criterions give quality indicators between 1 and N

$$ESS_k \approx \left(\sum_{i=1}^{N} (W_k^{(i)})^2 \right)^{-1}$$

Particle methods

- They are more appropriate than MCMC in several situations (highly correlated variables, multimodality)
- Do not require convergence time to equilibrium, suitable for dynamic estimation problems
- But: no "black box" software allowing the use of these techniques by non-experts

Summary

BiiPS software

Objectives

Develop a "black box" software to make **B**ayesian **i**nference using **i**nteracting **P**article **S**ystems.

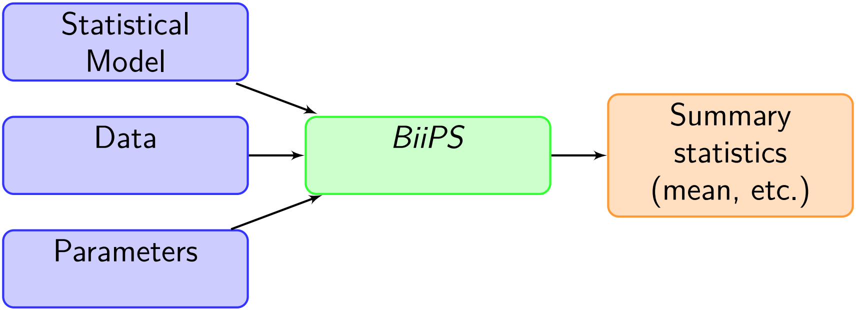


Figure: Input/Output diagram

BiiPS software

Features

- Core libraries in C++ (> 25K lines of code) making use of Boost
  - Creates a graphical model and executes a particle algorithm (filtering and smoothing)
  - Selects automatically the order of the variables to be sampled
  - Selects automatically the laws of exploration (conjugate and non conjugate cases)
  - Module with its extensible set of functions, distributions and samplers
- BUGS language compatible: compiler adapted from JAGS © M. Plummer
- RBiips interface to R making use of Rcpp package

Example

Financial econometry

Consider infering the underlying volatility $X_{1:t}$ from observed price or rate data $Y_{1:t}$

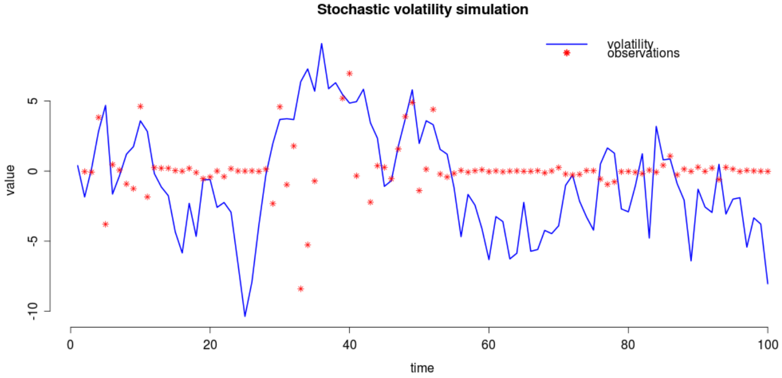
$$X_1 \sim \mathcal{N}(0, \frac{\sigma^2}{1-\alpha^2})$$
$$X_t|X_{t-1} \sim \mathcal{N}(\alpha x_{t-1}, \frac{\sigma^2}{1-\alpha^2}) \quad t > 1$$
$$y_t|X_t \sim \mathcal{N}(0, \beta^2 \exp(x_t)) \quad t > 1$$

BUGS language "volatility.bug"

```
model {
  prec.x <- (1-alpha^2) / sigma^2
  x[1] ~ dnorm(0, prec.x)
  for (t in 2:t.max) {
    x[t] ~ dnorm(alpha * x[t-1], prec.x)
    prec.y[t] <- 1 / (beta^2 * exp(x[t]))
    y[t] ~ dnorm(0, prec.y[t])
  }
}
```

Example

Financial econometry



Example

Financial econometry

```
# Define data
data <- list(t.max=100, sigma=1.0,
             alpha=0.91, beta=0.5,
             y=y)
```

Example

Financial econometry

```
# Define data
data <- list(t.max=100, sigma=1.0,
             alpha=0.91, beta=0.5,
             y=y)

# Compile the model and load the data
model <- biips.model("volatility.bug",
                      data)
```

Example

Financial econometry

```
# Define data
data <- list(t.max=100, sigma=1.0,
             alpha=0.91, beta=0.5,
             y=y)

# Compile the model and load the data
model <- biips.model("volatility.bug",
                     data)

# Run SMC algorithm
out.smc <- smc.samples(model, "x",
                       n.part=1000)
```

Example

Financial econometry

```
# Summary statistics
x.summ <- summary(out.smc$x,
                  fun=c("mean","quantiles"),
                  probs=c(.05,.95))
plot(x.summ)
```

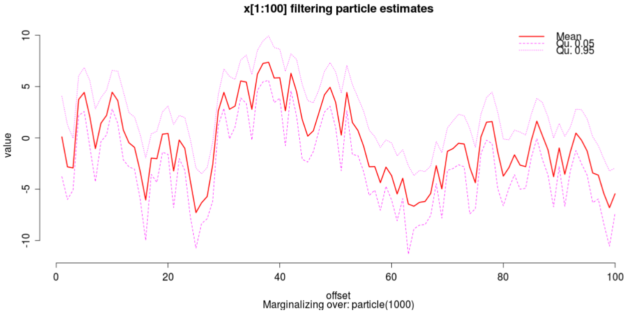


Figure: Summary statistics

Example

Financial econometry

```
# Kernel density estimates
plot(density(out.smc$x, adjust=2))
```

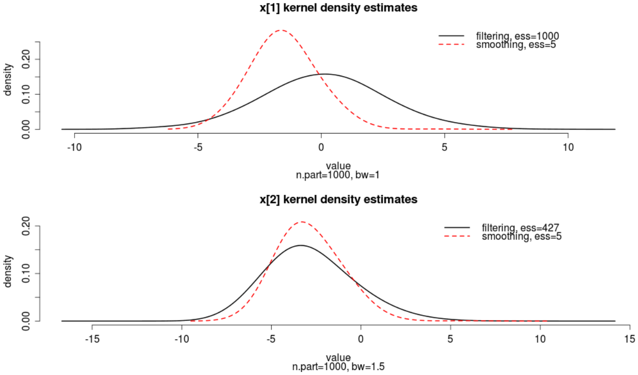


Figure: Kernel density estimates

Perspectives

Optimization

- Parallelization
- Reduce memory footprint

Software extensions

- More conjugate samplers, distributions and functions
- More advanced particle techniques
- Allow external user defined functions and samplers
- Interfaces: Matlab, Python, standalone

Summary

Conclusion

- BiiPS is a general software for Bayesian inference on graphical models
- Implements SMC/particle methods in a black box fashion
- Easy to use RBiips package

References

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THANK YOU

http://alea.bordeaux.inria.fr/biips

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